

# Langmuir waves dispersion in semi-relativistic spinless quantum plasma

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Based on the Darwin's Hamiltonian the many particle quantum hydrodynamics is considered. Force field appearing in the corresponding Euler equation is considered in details, contribution of different terms of the Darwin's Hamiltonian in the Euler equation is traced. For example, relativistic correction to particle kinetic energy leads to several terms in the Euler equation, these terms have different form, and one of them has form looking like a term appearing from the Darwin term. So, two different mechanisms give analogous contribution in wave dispersion, which differ by sign of terms. Microscopic analog of the Biot-Savart law, called current-current interaction, and describing interaction of moving charges via magnetic field, is also included in our description. The semi-relativistic generalization of the quantum Bohm potential is obtained. Contribution of relativistic effects in the spectrum of plasma collective excitations is considered.

## I. INTRODUCTION

There is fast growing of the interests to the theory of the relativistic [1]- [5] and semi-relativistic [6] quantum plasma. In this paper we develop the many-particle quantum hydrodynamics (QHD) [7]- [11] in semi-relativistic approximation, which allows successive analysis of many-particle effects. In this way we are going to discuss betweenness relation quantum, thermal, semi-relativistic effects in the system of many charged particles, as the result we will present complete theory including described effects for the spinless charged particles.

Spin caused effects appear in theory in the semi-relativistic approximation, however they play significant role in non-relativistic physical systems, such as, for example, ferromagnetic. Spin dynamics is also very important in physics of the quantum plasma, where electrons and positrons are most widespread objects and their spin is the inherent dynamical property. In last decade a lot of papers were dedicated to study of spin dynamics in quantum plasma, especially by means the quantum hydrodynamics and Vlasov-like kinetic equations. However, it is very interesting and important to understand the quantum many-particle physics appearing from consideration of the Darwin's Hamiltonian, which is the spinless analog of the Breit's Hamiltonian [12]. The Darwin's Hamiltonian contains both the non-relativistic terms, which describe particle kinetic energy and the Coulomb interaction, and semi-relativistic terms describing relativistic correction to particle kinetic energy (RCKE), which should be important at studying of relativistic beams in plasma, interaction energy of moving charges (which is also called current-current interaction and presents itself generalized Biot-Savart-Laplace law), and the term proportional to the Dirac's delta function called Darwin term or Darwin's interaction.

Suggestion were made in Ref. [6] that in some cases contribution of the RCKE much smaller than the Darwin term. However our studies of semi-relativistic effects in the quantum plasma and especially quantities entering the

Darwin's Hamiltonian, based on the quantum hydrodynamics method, shows that the RCKE leads to existence of a number of terms in the semi-relativistic Euler equation, which have various forms described below. One of the terms caused by the RCKE has form close to the only term brought by the Darwin term.

Contribution of the RCKE and current-current interaction in the plasma wave dispersion and the electron beam have recently been considered [13] in terms of the many-particle quantum hydrodynamics method developed in Ref.s [7]- [11]. Another method of derivation of the QHD equation for systems of charged spinning particles was suggested later in Ref.s [14], [15]. Some aspects of quantum plasma physics were reviewed in Ref. [16]. Before we pass to detailed description of the quantum model we admit that classic description of plasma theories can be found, for example, in Ref.s [17], [18]. New method of derivation of classic hydrodynamics equation from microscopic particles motion driven by Newton's equations is presented in Ref. [19]. This method does not include using of a kinetic equation as intermediate step of the derivation.

This paper is dedicated to comparison of contribution of the RCKE, the Darwin term and the current-current interaction in the Euler equation, obtaining of the explicit form of the semi-relativistic pressure tensor and it's influence on the dispersion properties of the longitudinal waves.

Our paper is organized as follows. In Sec. II we discuss basic Hamiltonian and compare contribution of different terms. In Sec. III In Sec. IV the method of obtaining of dispersion equation is described, linearized set of the semi-relativistic Euler equation is presented. In Sec. V dispersion dependence of quantum semi-relativistic Langmuir waves is calculated and discussed. In Sec. VI brief summary of obtained results is presented.

## II. THE MODEL DESCRIPTION

The equations of quantum hydrodynamic are derived from the non-stationary Schrodinger equation for system

of  $N$  particles:

$$i\hbar\partial_t\psi(R, t) = \hat{H}\psi(R, t)$$

with Hamiltonian

$$\begin{aligned} \hat{H} = & \sum_i \left( \frac{1}{2m_i} \mathbf{D}_i^2 - \frac{1}{8m_i^3 c^2} \mathbf{D}_i^4 \right. \\ & \left. + e_i \varphi_{i,ext} - \frac{e_i \hbar^2}{8m_i^2 c^2} \text{div} \mathbf{E}_{i,ext} \right) \\ & + \frac{1}{2} \sum_{i,j \neq i} \left( e_i e_j G_{ij} - \frac{\pi e_i e_j \hbar^2}{2c^2} \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} \right) \delta(\mathbf{r}_i - \mathbf{r}_j) \right. \\ & \left. - \frac{e_i e_j}{2m_i m_j c^2} G_{ij}^{\alpha\beta} \right). \end{aligned} \quad (1)$$

The following designations are used in the equation (1):  $e_i$ ,  $m_i$ -are the charge and the mass of particle,  $\hbar$ -is the Planck constant and  $c$  is the speed of light,  $D_i^\alpha = -i\hbar\partial_i^\alpha - e_i A_{i,ext}^\alpha/c$ ,  $\varphi_{i,ext}$ ,  $A_{i,ext}^\alpha$  - the potentials of the external electromagnetic field,  $\partial_i^\alpha = \nabla_i^\alpha$  is the spatial derivatives,  $G_{ij} = 1/r_{ij}$ , - is the Green functions of the Coulomb interaction,  $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ ,

$$G_{ij}^{\alpha\beta} = \frac{\delta^{\alpha\beta}}{r_{ij}} + \frac{r_{ij}^\alpha r_{ij}^\beta}{r_{ij}^3} \quad (2)$$

is the Green function of the current-current interaction,  $\psi(R, t)$ -is psi function of  $N$  particle system,  $R = (\mathbf{r}_1, \dots, \mathbf{r}_N)$ . Let's consider physical meaning of the terms in the Hamiltonian (1).

The first term in the right-hand side of the Hamiltonian is the kinetic energy, the second is the RCKE, the third is the potential energy of the classic charge in the external electric field, the fourth is the quantum contribution in the energy of the charge being in the external electric field, which is called the Darwin term. All these terms are valid for each particle, as they describe kinematic properties and interaction with external field. The next terms describe inter-particle interactions. First of all, by the fifth term, the Coulomb interaction is presented. The next term describes quantum contribution in the interaction of charges. It is the Darwin's interaction, corresponding to the Darwin term. And the last term describes the current-current interaction, which is the microscopic analog of the Biot-Savart law.

The fourth and sixth terms describe the Darwin term. The fourth term shows semi-relativistic contribution to the force acting from external electric field on charged particle. The sixth term presents Darwin term between two particles, which can be considered as a semi-relativistic addition to the Coulomb interaction. If we have deal with interaction of two electrons the Darwin term is

$$H_D = -\pi \left( \frac{e\hbar}{mc} \right)^2 \delta(\mathbf{r}_i - \mathbf{r}_j), \quad (3)$$

where  $\mathbf{r}_i$  and  $\mathbf{r}_j$  are the coordinates of the two electrons. The explicit form of the Darwin term was derived from quantum electrodynamics scattering amplitude. Now we have to compare the two Darwin terms describing interaction with external field, the fourth term in equation (1), which appearing in the semi-relativistic limit of the Dirac's equation [12], and inter-particle interaction, the sixth term in equation (1). Admitting what  $\Delta_i(1/|\mathbf{r}_i - \mathbf{r}_j|) = -4\pi\delta(\mathbf{r}_i - \mathbf{r}_j)$  and introducing microscopic electric field caused by, for example, particle  $j$  and acting on particle  $i$ , as  $\mathbf{E}_{ij} = -\nabla_i(e_j/r_{ij})$ . We see that sixth term in Hamiltonian (1) can be represented as

$$H_D = -\frac{e_i \hbar^2}{8c^2} \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} \right) \nabla_i \mathbf{E}_{ij}. \quad (4)$$

Comparing the fourth term in the Hamiltonian (1) and formula (4) we get that these terms coincide if  $m_2 \rightarrow \infty$ . However if we consider interaction of two electrons we have  $m_i = m_j$  and from (3) we get

$$H_D = -\frac{e\hbar^2}{4m^2 c^2} \nabla_i \mathbf{E}_{ij},$$

what differs in the two times from the fourth term in (1). Actually discussed terms should coincide due to the principle of field superposition, so we put additional factor two in the fourth term in the Hamiltonian (1), but we will keep in mind that we can make another choice and follow to the consequence of the Dirac's equation. At discussion of wave dispersion we will consider consequences of the both choices of the coefficient in the Darwin terms.

We have included Darwin's interaction in the Hamiltonian, and for short references below we introduce new function  $\tilde{G}_{ij}$  which is defined as  $\tilde{G}_{ij} = G_{ij} - (\hbar^2/4m^2 c^2)\delta(\mathbf{r}_i - \mathbf{r}_j)$ .

$\tilde{G}_{ij}$  leads to existence of two terms in the force field in the Euler equation. Let's consider how they emerge during derivation of the semi-relativistic Euler equation, what we make differentiating current  $\mathbf{j}$ , appearing in the continuity equation, with respect to time and using the Schrodinger equation. One of them appears due to commutation of  $\tilde{G}_{ij}$  with the momentum operator  $\hat{p}_i^\alpha$  in the current  $\mathbf{j}$ . Let's point out that the operator  $\hat{p}_i^\alpha$  exists in the current  $\mathbf{j}$  due to operator of the kinetic energy in the Hamiltonian (1). In the self-consistent field approximation this term has following form:

$$F_C^\alpha = -e^2 n \partial^\alpha \int d\mathbf{r}' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{\pi \hbar^2}{m^2 c^2} \delta(\mathbf{r} - \mathbf{r}') \right) n(\mathbf{r}', t). \quad (5)$$

The self-consistent field approximation allows us introduce electric field  $\mathbf{E}$  caused by the charges of the system, which has following explicit form

$$\mathbf{E} = -e\nabla \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} n(\mathbf{r}', t), \quad (6)$$

where  $n$  is the particle concentration, and satisfy to the quasi-electrostatic Maxwell's equations:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = 0 \quad (7)$$

$$\nabla \mathbf{E}(\mathbf{r}, t) = 4\pi \sum_a e_a n_a(\mathbf{r}, t), \quad (8)$$

where subindex "a" describe the species of particle: electrons, ions and positrons. In the case of the one species we can rewrite the last formula as

$$\nabla \mathbf{E}(\mathbf{r}, t) = 4\pi e n(\mathbf{r}, t). \quad (9)$$

Now, force field  $\mathbf{F}_C$  (5) take form

$$F_C^\alpha = e n E^\alpha - \frac{\pi e^2 \hbar^2}{m^2 c^2} n \partial^\alpha n, \quad (10)$$

where the concentration under the space derivative represents source of the field, which in the case of the first term has come to equation (9). So, using equation (9) for the mentioned concentration we come to

$$F_C^\alpha = e n E^\alpha - \frac{e^2 \hbar^2}{4m^2 c^2} n \partial^\alpha (\partial^\beta E^\beta). \quad (11)$$

Presented here form of the second term corresponds to the semi-relativistic contribution in the force acting on the charged particle from external electric field obtained from the Dirac's equation [12].

The second term caused by  $\tilde{G}_{ij}$  appears due to the RCKE which is a semi-relativistic term. In the self-consistent field approximation it appears as

$$F_{sr}^\alpha = \frac{e \hbar^2}{4m^2 c^2} \partial_\beta (\partial_\alpha E_\beta \cdot n). \quad (12)$$

As RCKE has semi-relativistic origin we can write  $G_{ij}$  instead of  $\tilde{G}_{ij}$  in this term. The RCKE also gives other terms in the force field, all of them are presented below in the Euler equation.

Equations (11) and (12) are very similar, they have two differences. The first difference is distinction in tensor structure and the second one is the fact that equation (12) contains concentration under the spatial derivative, whereas in formula (11) concentration contains as an external multiplier.

### III. QUANTUM HYDRODYNAMICS EQUATIONS

To trace contribution of different semi-relativistic terms in the wave dispersion we introduce dimensionless coefficients  $\theta$  and  $\xi$  equal to 1 and marking different terms. We put  $\theta$  in the second term of formula (11) and  $\xi$  in the formula (12).

The first equation of the QHD set is the continuity equation:

$$\partial_t n + \nabla \mathbf{j} = 0. \quad (13)$$

In that equation a function of current  $\mathbf{j}(\mathbf{r}, t) = n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)$  is arisen, where  $\mathbf{v}(\mathbf{r}, t)$  - is the velocity field.

Second equation of the QHD set is the Euler equation, but in the semi-relativistic approximation function  $\mathbf{j}(\mathbf{r}, t)$  appeared in the continuity equation in the current of particles, however in contrast to non-relativistic case we can not call it momentum density, thus the Euler equation is the equation of evolution of particles current [20]. This equation has form

$$\begin{aligned} mn(\partial_t + v^\beta \nabla^\beta) v^\alpha + \partial_\beta P_{\alpha\beta} &= enE^\alpha + \frac{e}{c} \varepsilon^{\alpha\beta\gamma} n v^\beta B^\gamma \\ &- \theta \frac{e \hbar^2}{4m^2 c^2} n \partial^\alpha (\partial^\beta E^\beta) + \xi \frac{e \hbar^2}{4m^2 c^2} \partial_\beta (\partial_\alpha E_\beta \cdot n) \\ &- \frac{e}{mc^2} \left[ E_\beta (mn v_\alpha v_\beta + P_{\alpha\beta}) + E_\alpha \left( \frac{1}{2} mn v^2 + n\varepsilon \right) \right] \\ &- \frac{e^2 \hbar^2}{8m^2 c^2} \partial_\beta n \int d\mathbf{r}' \partial_\alpha G_{\beta\gamma}(\mathbf{r} - \mathbf{r}') \partial'_\gamma n(\mathbf{r}', t) \\ &- \frac{e^3}{2mc^2} n \int d\mathbf{r}' G_{\alpha\beta}(\mathbf{r} - \mathbf{r}') E_\beta(\mathbf{r}', t) n(\mathbf{r}', t) \\ &+ \frac{e^2}{2c^2} \int d\mathbf{r}' [\partial_\alpha G_{\beta\gamma}(\mathbf{r} - \mathbf{r}') - \partial_\beta G_{\alpha\gamma}(\mathbf{r} - \mathbf{r}')] \pi_{\beta\gamma}(\mathbf{r}, \mathbf{r}', t) \\ &+ \frac{e^2}{2mc^2} n \int d\mathbf{r}' \partial_\gamma G_{\alpha\beta}(\mathbf{r} - \mathbf{r}') \times \\ &\times [mn(\mathbf{r}', t) v_\beta(\mathbf{r}', t) v_\gamma(\mathbf{r}', t) + P_{\beta\gamma}(\mathbf{r}', t)], \quad (14) \end{aligned}$$

where  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic fields,  $\varepsilon$  is the density of thermal energy, including quantum part, an analog of the quantum Bohm potential,  $\varepsilon^{\alpha\beta\gamma}$  - is the antisymmetric symbol (the Levi-Civita symbol),  $P^{\alpha\beta}$  is the tensor of pressure, which is the semi-relativistic generalization of the sum of non-relativistic thermal pressure  $p^{\alpha\beta}$  and the quantum Bohm potential  $T^{\alpha\beta}$ ,  $n\varepsilon$  is the density of the thermal energy. In right-hand side of equation (14) a force field locates. The force field consists of the Lorentz force, and specific quantum semi-relativistic terms, which will be discussed below.  $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$  will be presented explicitly and considered below after analysis of the structure of  $P^{\alpha\beta}$ . The vector potential of magnetic field appears in the form

$$A_\alpha^{int}(\mathbf{r}, t) = \frac{e}{2c} \int d\mathbf{r}' G_{\alpha\beta}(\mathbf{r} - \mathbf{r}') n(\mathbf{r}', t) v_\beta(\mathbf{r}', t), \quad (15)$$

which gives contribution in the Lorentz force, the second term in right-hand side of equation (14) along with external magnetic field. Magnetic field caused by currents  $\mathbf{B} = \nabla \times \mathbf{A}^{int}$  satisfies to the quasi-magnetostatic Maxwell's equation:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}, \quad (16)$$

and

$$\nabla \mathbf{B} = 0. \quad (17)$$

Before discussion of pressure tensor  $P^{\alpha\beta}$  we going to discuss the physical meaning of terms of the force field presented in the right-hand side of the Euler equation (14). It is especially important as some of these terms presented for the first time. Thus, the first two terms present the density of the Lorentz force. The self-consistent part of the Coulomb interaction gives contribution in the first term. The second term contains contribution of the current-current interaction in the self-consistent field approximation. We already should admit that all terms in the Euler equation are presented in the self-consistent field approximation. Actually, only part of the whole contribution of the current-current interaction came in the Lorentz force, it also leads to several other terms. They are sixth–ninth terms of the Euler equation. In fact, the terms seven–nine already appear in the classic semi-relativistic hydrodynamics (detailed discussion of the classic theory will be presented in [13]), but in the quantum theory these terms have more rich structure. First of all they contain contribution of the exchange interactions via quantum correlations, which do not considered in this paper, but they naturally appear in the many-particle QHD, and we neglect them here, considering the self-consistent field approximation only. The third term caused by the Darwin's interaction, it includes contribution of both the inter-particle Darwin's interaction and the Darwin term describing action of external electric field. The terms four and five present contribution of the RCKE. The fourth term has simple structure, it contains divergence  $\nabla^\beta$  of the tensor which is product of particles concentration on  $\nabla^\alpha E^\beta$ . In the fifth term, which contains a number of terms in square brackets, the first set of them is the convolution of  $E^\beta$  with the tensor which is the current of the particles current  $\mathbf{j}$ , as a part of this current we have the pressure tensor  $P^{\alpha\beta}$ . As the fifth term of the Euler equation has semi-relativistic nature we should consider non-relativistic part of  $P^{\alpha\beta}$ . The second set of terms in the fifth term is the product of electric field  $E^\alpha$  on the energy density. The energy density was separated on two parts there. First of them is the density of the kinetic energy of a local ordered motion. We need to say that  $n\varepsilon$  is the energy density which consists of two parts: thermal energy and quantum contribution – an analog of the quantum Bohm potential. In one particle case we lose contribution of thermal motion and quantum-thermal terms, and we get quantum terms arising for non-interacting particles.  $\varepsilon$  gives no contribution in considered below problem, therefore we do not present it's explicit form.

Comparing the third and the fourth terms in equation (14) we find that in the linear approximation at  $\xi = \theta$  they reduce each other, or at  $\xi \neq \theta$  they common contribution is less than each taken separately, and sign of common contribution depend on relation between  $\xi$  and  $\theta$ . Consequently, simultaneous consideration of the RCKE and Darwin term is necessary. The seventh term appears because

of the simultaneous account of the Coulomb and current-current interactions. It is an analog of the term four, each of them appears in the equation due to commutation of corresponding terms in the current  $\mathbf{j}$  with the Hamiltonian of the Coulomb interaction and energy of charge interaction with the external electric field.

Explicit form of the tensor  $P^{\alpha\beta}$  is

$$\begin{aligned} P_{\alpha\beta}(\mathbf{r}, t) = & \int dR \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) a^2 \times \\ & \times [m u_{i\alpha} u_{i\beta} - \frac{\hbar^2}{2m} \left(1 - \frac{v_i^2}{c^2}\right) \partial_{i\alpha} \partial_{i\beta} \ln a \\ & + \frac{\hbar^2}{2mc^2} (\partial_{i\alpha} v_{i\gamma} \partial_{i\beta} v_{i\gamma} + v_{i\gamma} \partial_{i\alpha} \partial_{i\beta} v_{i\gamma}) \\ & + \frac{\hbar^2}{4mc^2} (v_{i\alpha} \partial_{i\beta} + v_{i\beta} \partial_{i\alpha}) (\partial_{i\gamma} v_{i\gamma} + 2v_{i\gamma} \partial_{i\gamma} \ln a) \\ & - \frac{\hbar^4 a^{-2}}{4m^3 c^2} \left( a \partial_{i\alpha} \partial_{i\beta} \Delta_i a + \partial_{i\alpha} \partial_{i\beta} a \Delta_i a \right. \\ & \left. - \partial_{i\alpha} a \partial_{i\beta} \Delta_i a - \partial_{i\beta} a \partial_{i\alpha} \Delta_i a \right) \\ & + \int dR \sum_{i=1, j=1, i \neq j}^N \delta(\mathbf{r} - \mathbf{r}_i) a^2 \frac{\hbar^2 e^2}{4m^2 c^2} \times \\ & \times (G_{ij}^{\beta\gamma} \partial_{i\alpha} \partial_{j\gamma} \ln a + G_{ij}^{\alpha\gamma} \partial_{i\beta} \partial_{j\gamma} \ln a), \quad (18) \end{aligned}$$

where  $v_{i\alpha}$  is the velocity of  $i$ -th particle, and it is the sum of the velocity field  $v(\mathbf{r}, t)$  and thermal velocity  $u_i$ ,  $a$  is the amplitude of the wave function  $\psi(R, t) = a \exp(iS/\hbar)$ , velocity of  $i$ -th particle  $v_{i\alpha}$  connects with the phase of the wave function as  $v_{i\alpha} = \partial_{i\alpha} S/m$ . The first term in formula is the non-relativistic thermal pressure, the second term in this formula consist of two parts, the first of them is the non-relativistic quantum Bohm potential, the other terms present semi-relativistic effects. Neglecting thermal velocities in the semi-relativistic terms of the pressure tensor  $P_{\alpha\beta}$  we get purely quantum semi-relativistic pressure, which is the semi-relativistic generalization of the quantum Bohm potential  $T_{\alpha\beta}$ , which explicit form for ideal gas is

$$T_{\alpha\beta} = -\frac{\hbar^2}{4m} \partial^\alpha \partial^\beta n + \frac{\hbar^2}{4m} \left( \frac{\partial^\alpha n \cdot \partial^\beta n}{n} \right). \quad (19)$$

We also drop contribution of the current-current interaction. In the result we have

$$\begin{aligned} P_{\alpha\beta}(\mathbf{r}, t) = & p_{\alpha\beta} + T_{\alpha\beta} - \frac{v^2}{c^2} T_{\alpha\beta} \\ & + \frac{\hbar^2}{2mc^2} n (\partial^\alpha v^\gamma \partial^\beta v_\gamma + v^\gamma \partial^\alpha \partial^\beta v_\gamma) \end{aligned}$$

$$\begin{aligned}
& + \frac{\hbar^2}{4mc^2} n(v^\alpha \partial^\beta + v^\beta \partial^\alpha)(\nabla \mathbf{v}) \\
& + \frac{\hbar^2}{4mc^2} (\partial^\gamma n) \left( v^\alpha \partial^\beta v^\gamma + v^\beta \partial^\alpha v^\gamma \right) \\
& - \frac{1}{c^2} \left( v^\alpha v^\gamma T^{\beta\gamma} + v^\beta v^\gamma T^{\alpha\gamma} \right) \\
& - \frac{\hbar^4}{4m^3 c^2} \left( \sqrt{n} \cdot \partial^\alpha \partial^\beta \Delta \sqrt{n} + \partial^\alpha \partial^\beta \sqrt{n} \cdot \Delta \sqrt{n} \right. \\
& \left. - \partial^\alpha \sqrt{n} \cdot \partial^\beta \Delta \sqrt{n} - \partial^\beta \sqrt{n} \cdot \partial^\alpha \Delta \sqrt{n} \right). \quad (20)
\end{aligned}$$

the first two terms have non-relativistic nature. The other terms are semi-relativistic, most of them are proportional to  $v^2(\mathbf{r}, t)/c^2$ , except of the four last terms. Thermal pressure  $p_{\alpha\beta}$  does not depend on interaction, so we can use equation of state for ideal gas, and we write  $p^{\alpha\beta} = nk_B T \delta^{\alpha\beta}$ , where  $k_B$  is the Boltzmann constant,  $T$  is the temperature,  $\delta^{\alpha\beta}$  is the Kronecker symbol. When  $P^{\alpha\beta}$  stays in a semi-relativistic term we should neglect semi-relativistic part and consider non-relativistic one (19) only.

Here we present explicit form of  $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$ , which is the part of the seventh term in the force field in the Euler equation

$$\begin{aligned}
\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) &= \int \prod_{j=1}^N d\mathbf{r}_j \sum_{i,j=1, i \neq j}^N \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \times \\
&\times a^2 (u_{i\alpha} u_{j\beta} - \frac{\hbar^2}{2m^2} \partial_{i\alpha} \partial_{j\beta} \ln a),
\end{aligned}$$

to close the QHD set of equations we should find approximate connection between  $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$  and other hydrodynamic quantities. Calculating  $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$  for the system of independent particles we get  $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = 0$ . Thus, in the first approximation we do not need to account contribution of  $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$  in the QHD equations.

#### IV. DISPERSION EQUATION OF QUANTUM SEMI-RELATIVISTIC LANGMUIR WAVES

To get semi-relativistic effects in the form of analytic simple formulas we consider quantum motion of electrons on the background of motionless ions. We consider the small perturbation of equilibrium state like

$$n_e = n_{0e} + \delta n_e, \quad \mathbf{v}_e = 0 + \mathbf{v}_e,$$

$$\mathbf{E} = 0 + \delta \mathbf{E}, \quad \mathbf{B} = 0 + \delta \mathbf{B}, \quad \delta p_e = 3mv_{se}^2 \delta n,$$

where  $m$  is the mass of the electron. In equations (13), (14) and the Maxwell's equations (7), (8), (16) and (17),  $v_{se}^2$  is the thermal velocity, for the case of degenerate electrons  $v_{se}^2$  is the Fermi velocity. Substituting these relations

into the system of equations and neglecting by nonlinear terms, we obtain a system of linear homogeneous equations in partial derivatives with constant coefficients. Passing to the following representation for small perturbations  $\delta f$

$$\delta f = f(\omega, \mathbf{k}) \exp(-i\omega t + i\mathbf{k}\mathbf{r})$$

yields a homogeneous system of algebraic equations.

The Euler equation (14) is very complicated, thus we allow ourself to present the algebraic form of linearized Euler equation

$$\begin{aligned}
& -i\omega m n_0 \delta v^\alpha + ik^\alpha \left( 3mv_{se}^2 + \frac{\hbar^2 k^2}{4m} - \frac{\hbar^4 k^4}{8m^3 c^2} \right) \delta n \\
& = en_0 \delta E^\alpha + k^\alpha k^\beta (\theta - \xi) \frac{en_0 \hbar^2}{4m^2 c^2} \delta E^\beta - \frac{e^3 n_0^2}{2mc^2} G^{\alpha\beta}(\mathbf{k}) \delta E^\beta, \quad (21)
\end{aligned}$$

where  $G^{\alpha\beta}(\mathbf{k})$  is the Fourier image of the Green function of the current-current interaction (2), it's explicit form is

$$G^{\alpha\beta}(\mathbf{k}) = \frac{8\pi}{k^2} \left( \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right).$$

The last term in the Euler equation (14) give a linear term due to the linear part of the quantum Bohm potential, which is a part of  $P^{\beta\gamma}$ , but it is equal to zero because of the tensor structure of  $G^{\alpha\beta}(\mathbf{k})$ . The last term in equation (21) gives no contribution in the dispersion of the Langmuir waves.

The electric field strength is assumed to have a nonzero value. Expressing all the quantities entering the system of equations in terms of the electric field, we come to the equation

$$\begin{aligned}
\omega^2 &= \omega_{Le}^2 \left( 1 + (\theta - \xi) \frac{\hbar^2 k^2}{4m^2 c^2} \right) \\
&+ \left( 3v_{se}^2 + \frac{\hbar^2 k^2}{4m^2} - \frac{\hbar^4 k^4}{8m^4 c^2} \right) k^2, \quad (22)
\end{aligned}$$

where  $\omega_{Le}$  is the Langmuir frequency,  $\omega_{Le}^2 = 4\pi e^2 n_0 / m$ . The first term in the right-hand side of (22) consists of three parts: the Langmuir frequency, contribution of the Darwin's interaction, proportional to  $\theta$ , and the third is contribution of the fourth term of the Euler equation, caused by the RCKE. The second group in equation (22) also consists of three parts: contribution of the pressure (thermal motion or Fermi pressure), the next part is the well-known quantum Bohm potential, the last part is the contribution of the RCKE via the semi-relativistic part of the tensor of pressure. In the next section we will discussed properties of the obtained dispersion dependence.

## V. DISPERSION DEPENDENCE OF LANGMUIR WAVES

In this section we will discuss contribution of term proportional to  $\theta - \xi$  in the dispersion of the Langmuir waves. Other term have the same form as presented in the previous section.

Following to the Breit's Hamiltonian we put  $\theta = 1$ , and according to formula (12) we also put  $\xi = 1$ . Consequently we have  $\theta - \xi = 0$ . In the results we have no contribution of the Darwin term in the Langmuir wave dispersion, since it compensates by one of the terms caused by RCKE, thus we have

$$\omega^2 = \omega_{Le}^2 + \left( 3v_{se}^2 + \frac{\hbar^2 k^2}{4m^2} - \frac{\hbar^4 k^4}{8m^4 c^2} \right) k^2. \quad (23)$$

However, we can choose  $\theta = 1/2$  if we follow to the semi-relativistic approximation of the Dirac's equation, and we have  $\xi = 1$ , which is only choice for  $\xi$ . In the result we get

$$\omega^2 = \omega_{Le}^2 \left( 1 - \frac{\hbar^2 k^2}{8m^2 c^2} \right) + \left( 3v_{se}^2 + \frac{\hbar^2 k^2}{4m^2} - \frac{\hbar^4 k^4}{8m^4 c^2} \right) k^2, \quad (24)$$

where we find the term arising in the balance of the RCKE and the Darwin term. If we do not consider contribution of RCKE, i.e. we put  $\xi = 0$ , we will get an analog of the formula (24), but with sign plus instead of minus.

## VII. CONCLUSION

We gave derivation of the many-particles QHD equations for the semi-relativistic system of spinless charged particles. We got contribution of the RCKE, the Coulomb, Darwin and current-current interactions in the Euler equation is obtained. Comparison of the different terms contributions is made. It is showed that simultaneous account of the RCKE and Darwin term is necessary, because the RCKE gives a number of terms having different structure, and one of them has structure of the term describing the Darwin term. The RCKE leads to the complex structure of the pressure tensor. The semi-relativistic part of the pressure tensor contains terms proportional to the rate of the velocity field to the square of light, and also includes the several terms proportional to  $\hbar^4/c^2$  and contain more higher spatial derivative than in the non-relativistic quantum Bohm pressure.

Using developed approximation of the QHD equations we studied dispersion dependence  $\omega(\mathbf{k})$  of the semi-relativistic Langmuir waves. We got contribution of the RCKE, which gives two term in  $\omega(\mathbf{k})$ , and the Darwin term, which gives one term. However, only one term emerges in the  $\omega(\mathbf{k})$ , since one of the terms caused by the

RCKE cuts the term caused by the Darwin term, so sum of them equal to zero. It is valid if we follow to the Breits Hamiltonian in choosing of the coefficient's for the Darwin term. In the other case, this term does not equal to zero. We discuss how different choice of the coefficient's for the Darwin term reveals in the  $\omega(\mathbf{k})$ .

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